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# Equilibrium of Spatial Structures Using 3-D Reciprocal Diagrams

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**Summary:** Based on Rankine's proposition for equilibrium of polyhedral frames in 1864, this research provides a clear illustration of the theory of reciprocity between form and force diagrams in three dimensions. It explores the geometric relationship between three-dimensional networks to determine the equilibrium of general spatial frames. It extends graphic statics to a fully three-dimensional method to design and analyse spatial frames such as tree structures, curved frames, cellular structures, etc., under non-parallel, external loads or self-stressed, compression- or tension-only conditions.

**Keywords:** *Spatial equilibrium networks, three-dimensional reciprocity, graphical statics in three dimensions, equilibrium of polyhedral networks and branching structures*

## 1. INTRODUCTION

This paper presents a method for describing the equilibrium of polyhedral frames starting from a given force polyhedron. It is the first step in the generalization of graphic statics to fully three-dimensional problems. Graphic statics is a technique for structural design and analysis that solves the static equilibrium of structures by means of geometric construction. It is an important tool for architects and engineers who try to unify design with analysis to improve the *economics of construction* [1] and aesthetics of design.

The key feature of graphic static is the geometric relationship between the shape of a structure and the corresponding system of forces, represented by *form* and *force* diagrams. This relationship enables designers to observe the changes in one diagram as a result of a modification of the other, and provides them with direct control over the form and the forces of structural systems.

The theoretical framework for graphic statics was established by Rankine [2] and Maxwell [3]. Rankine proposed the idea of reciprocity between the form of the structure and the diagram of forces. Maxwell provided a geometrical procedure for drawing these reciprocal diagrams. Since then, many people have contributed to the further development of their framework, but graphic statics as we know it today is often attributed to the contributions of Culmann [4], who gathered the mathematical proofs of a projective relationship between a funicular polygon and its force polygon, introduced earlier by Pierre Varignon [5], and Cremona who formulated graphic statics as a series of recipes for dealing with specific structural problems [6].

### 1.1. Problem statement

Today, by means of available computational tools, architects and engineers explore the geometric possibilities of design more than ever. Meanwhile, the arguments of structural efficiency, unity of design and analysis, and the economy of construction remain valid. Therefore, many designers are looking for tools that provide them with control over both the shape of a complex three-dimensional structure and the forces in it.

Holistic approaches to structural design, such as graphic statics, seem particularly suited for this purpose. However, existing methods in graphic statics cannot deal with three-dimensional problems, with the exception of a few limited developments only applicable to very specific problems (see e.g. [7]). Therefore, in the past, structural designers have addressed such problems by decomposing them into a series of equivalent 2-D problems, or, in some cases, by treating horizontal and vertical equilibrium separately [8].

### 1.2. Objectives

The overall objective of this research is to extend graphic statics to three-dimensional problems, thereby providing the basis for tools that allow (structural) designers to address the geometrical and structural challenges of contemporary projects in an integrated and efficient manner. In this paper specifically, as the first step, we will describe a method for exploring the equilibrium of spatial frames starting from a given (or chosen) force polyhedron.

The outline of the paper is as follows. In Section 2, we revisit and visualize the fundamental proposition for the equilibrium of polyhedral frames by Rankine, and discuss Maxwell's geometric solution for specific cases. In Section 3, we establish some definitions that form the basis for our approach. Section 4 of this paper includes an overview of the computational implementation of the procedure for finding the reciprocal polyhedral frame from a given force polyhedron, as well as a more detailed description of the major steps in the process. Section 5 shows how the presented approach can be used for form finding of tree structures, spatial frames, and cellular structures, starting from a given/chosen force distribution.

## 2. RECIPROCAL FIGURES IN THREE DIMENSIONS

In this section, we revisit the theory of reciprocity between the form and the force diagram proposed by Rankine, as well as the geometric procedure of constructing such reciprocal figures for specific cases, described by Maxwell's.

### 2.1. Rankine's principle of equilibrium of polyhedral frames

Rankine proposed the idea of reciprocity between form and force diagrams in his '*Principles of the Equilibrium of Polyhedral Frames'* [2]. He states that forces acting on a point perpendicular and proportional to the areas of the faces of a polyhedron are in equilibrium. The theory is limited to a short paragraph, and, to the knowledge of the authors, there is no complementary illustration. Since the text is not very well known and not readily available, and since it is short, we repeat it here as it appeared in the Philosophical Magazine:

*"If planes diverging from a point or line be drawn normal to the lines of resistance of the bars of a polyhedral frame, then the faces of a polyhedron whose edges lie in those diverging planes (in such a manner that those faces, together with the diverging planes which contain their edges, form a set of contiguous diverging pyramids or wedges) will represent, and be normal to, a system of forces which, being applied to the summits of the polyhedral frame, will balance each other- each such force being applied to the summit of meeting of the bars whose lines of resistance are normal to the set of diverging planes that enclose that face of the polyhedron of forces which represents and is normal to the force in question. Also the areas of the diverging planes will represent the stresses along the bars to whose lines of resistance they are respectively normal.*

*It is Obvious that the polyhedron of forces and the polyhedral frame are reciprocally related as follows: their numbers of edges are equal, and their corresponding pairs of edges perpendicular to each other; and the number of faces in each polyhedron is equal to the number of summits in the other."*

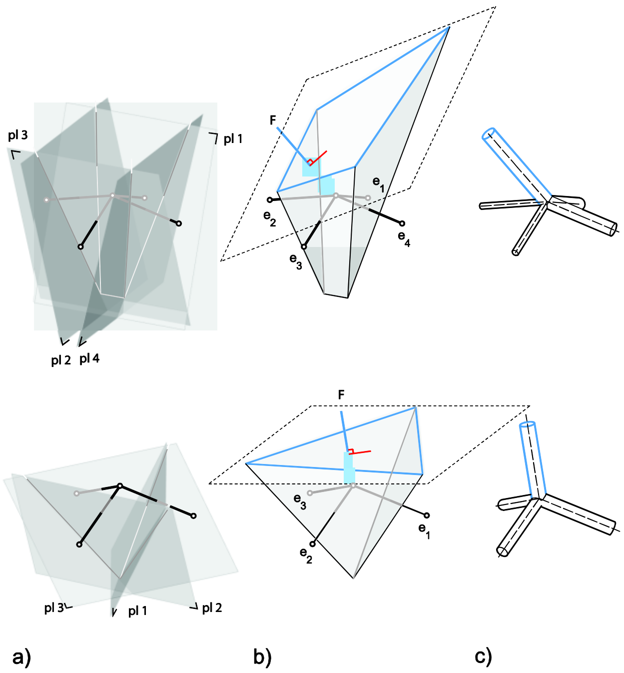


Figure 1. Two polyhedral frames with their reciprocal polyhedrons: a) planes perpendicular to each bar diverge from a line (top), or a point (bottom), their intersection forming an open polyhedron; b) The plane normal to the direction of an applied, external force closes the polyhedron and creates equilibrium; and c) The pipe diagram represents the magnitude of each force, calculated from the area of the corresponding (perpendicular) face in the force polyhedron.

Figure 1 visualizes and, thereby, clarifies the contents of Rankine's impressively dense paragraph. It depicts the application of his theory to two polyhedral frames, consisting of three and four bars, respectively. Figure 1.a represents the planes that are perpendicular to the bars of the frames, and diverge from a line (top) or a point (bottom). In both cases, the planes make a polyhedron with an open face. The face perpendicular to an additional force (for example, an applied load) then closes the polyhedron (Fig. 1.b), and creates force equilibrium for the node. Note that each choice for the closing face results in a different distribution of forces in the frame. The areas of the faces of the closed polyhedron are proportional to the magnitudes of the forces in the corresponding, perpendicular bars of the frame (Fig. 1.c).

### 2.2. Maxwell's reciprocity in three dimensions

Rankine, however, did not provide a method by which a polyhedral frame and its reciprocal force polyhedron may be constructed. Maxwell proposed to address this problem in a purely geometrical manner and stated some of the properties of reciprocal figures, and the condition of their existence [3].

According to Maxwell's (geometric) definition, reciprocal figures both consist, solely, of closed reciprocal polyhedrons such that:

i. each figure is made up of closed polyhedrons with planar faces;

ii. every point of intersecting lines in one figure is represented by a closed polyhedron in the other; and

iii. each face in each figure belongs to two and only two polyhedrons.

According to Maxwell, the simplest figure in space that fits this definition and for which thus a reciprocal can be found, is the group of tetrahedral cells resulting from five points in space (Fig. 2.a). These five points are connected with ten lines, which form ten triangular faces, which make up five tetrahedrons. Each face of this figure is shared by only two tetrahedrons (Fig. 2.b).

The reciprocal of this figure can be found through strictly geometric operations. Indeed, connecting the centres of the circumscribing spheres of each tetrahedron results in a figure in which the edges are perpendicular to the faces of the original figure (Fig. 2.c-d). Maxwell mentions that these reciprocal figures are the same as the reciprocal figures of Rankine, and calls one the force figure and the other the form figure. He furthermore states that, indeed, a mechanical interpretation of the relationship between these figures is that the area of a face in the force figure represents the magnitude of force in the line perpendicular to that face in the form figure such that the entire system is in equilibrium. For instance, in Figure 3, the area of the face that is made up by the three vertices $v'_2$, $v'_4$ and $v'_5$, is proportional to the magnitude of the force in edge $e_{21}$.

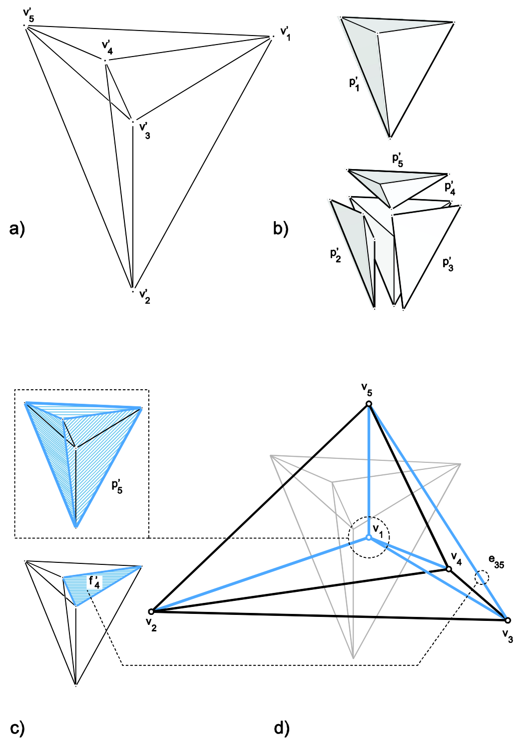


Figure 2. a) Five points in space connected by ten lines; b) Ten triangular faces and five tetrahedrons including the external tetrahedron and four internal tetrahedral cells; c) Force tetrahedrons d) Form tetrahedrons resulted from connecting the circumscribing spheres.

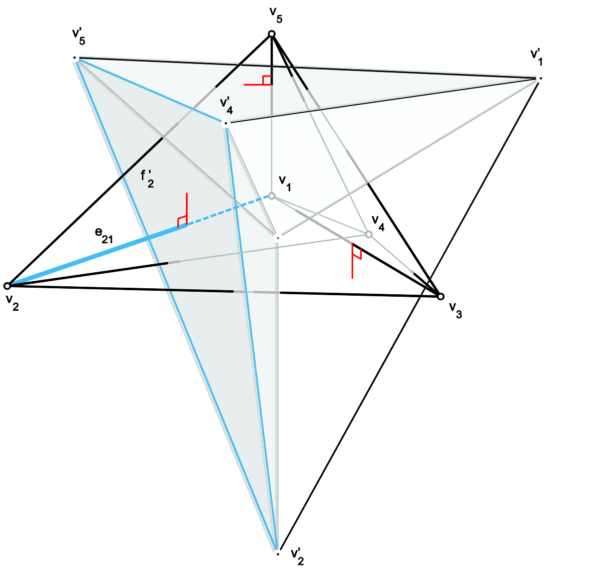


Figure 3. The area of the face between vertices $v'_2$, $v'_4$, $v'_5$, in the force figure is proportional to the magnitude of the force in $e_{21}$ in the form figure.

## 3. FORM AND FORCE POLYHEDRONS

In this section, we define the form polyhedron and the force polyhedron that allow more general structural systems to be modelled. Then, we describe how this definition can be used to control the equilibrium of both statically determinate and indeterminate structures.

### 3.1. Structural reciprocity in three dimensions

Although Maxwell provides an elegant solution for the specific cases delimited by his definition of reciprocity, his method cannot be applied to general polyhedral frames; for example, it does not allow the inclusion of external forces. In our definition, the form and the force polyhedrons have the following characteristics:

i. the force polyhedron consists of a (group of) closed polyhedron(s), representing and assuring equilibrium of the form polyhedron;

ii. the form polyhedron is an open spatial network, which allows the representation of external loads;

iii. each edge in the form polyhedron is perpendicular to its corresponding face in the force polyhedron. The magnitude of the force in that edge is proportional to the area of the face in the force polyhedron; and

iv. each node which is the intersection of at least four edges in the form polyhedron corresponds to a closed cell in the force polyhedron.

Here, we define the related notations that will be used throughout the paper. We label the elements of the form polyhedron with lowercase letters: $e_i$, $v_i$, $f_i$ and $p_i$ denote the $i$th edge, vertex, face and polyhedral cell, respectively. We name the elements of the force polyhedron in the same way, but suffixed with a prime symbol: $e'_i$, $v'_i$, $f'_i$ and $p'_i$.

Edge $e_i$ of the form polyhedron is reciprocal to face $f'_i$ of the force polyhedron. Additionally, vertex $v_i$ of the form polyhedron is reciprocal to polyhedral cell $p_i$ of the force polyhedron.

In accordance with these definitions , the simplest force polyhedron in space is a single tetrahedron. It represents the equilibrium of a single node in 3-D with four applied forces, as seen in Figure 4. The number of faces of the tetrahedron is equal to the number of forces applied to the node. Each edge, $e_i$, in the form polyhedron is perpendicular to its corresponding face, $f'_i$, in the force polyhedron, and the magnitude of force, $F_i$, along the edge, $e_i$, in the form polyhedron is equal to the area of the perpendicular face, $f'_i$, in the force polyhedron. Next, we will discuss how force polyhedrons can represent determinate and indeterminate systems of forces in space.

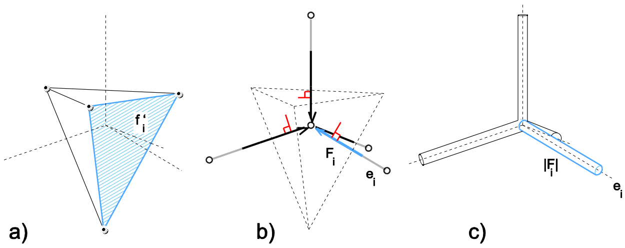


Figure 4. Equilibrium of a single node in space: a) a single tetrahedron as force polyhedron; b) its reciprocal form polyhedron consisting of four lines and the applied forces; c) a piped representation of the magnitude of the applied forces proportional to the areas of the faces of the tetrahedron.

### 3.2. Statically determinate system of forces in 3-D

Any aggregation of force tetrahedrons represents the equilibrium of a statically determinate spatial bar-node structure with applied loads and reaction forces. Uniformly scaling the force polyhedrons will change the magnitude of the forces in the system without changing their distribution or resulting in a frame with different geometry (Fig. 5).

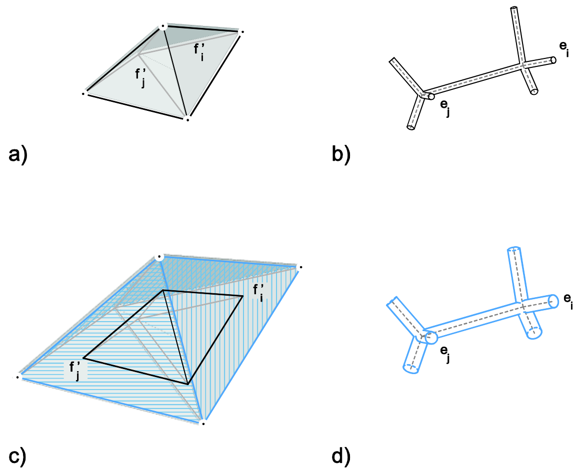


Figure 5. a) A force polyhedron consisting of two tetrahedrons; b) The reciprocal form polyhedron with the magnitude of forces shown as pipes; c) Scale transformation of the force polyhedron; d) Updated magnitude of forces.

### 3.3. Statically indeterminate system of forces in 3-D

Reciprocal form and force polyhedrons can also represent a possible equilibrium of an indeterminate system of forces. For example, Figure 6 depicts a system with force polyhedrons that are cubes. For the same geometry of the form polyhedron, changing the area of some of the faces in the force polyhedron changes the force distribution in the form polyhedron while allowing its geometry to stay fixed.

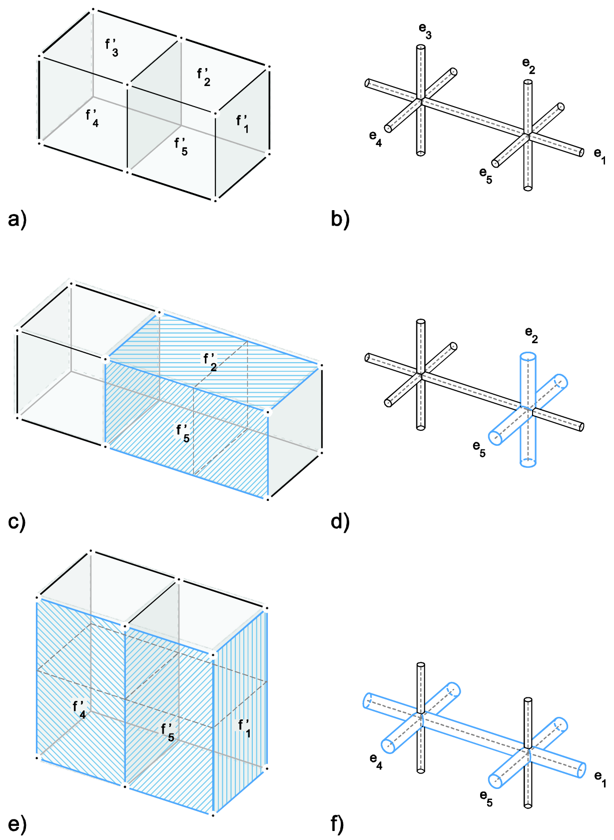


Figure 6. a) A force polyhedron consisting of two cubes; b) The reciprocal form polyhedron with the magnitude of forces shown as pipes. Transformations c) and e) maintain the reciprocal relation between force faces and form edges, and therefore result in different force distributions for the same form polyhedron, as shown in d) and f).

## 4. IMPLEMENTATION

In this section we describe how the reciprocal form polyhedron can be constructed from a given force polyhedron. The algorithm requires as input a set of connected lines representing the form polyhedron. This set should be such that each line is connected to at least two other lines at both ends, and such that the represented polyhedron has planar faces.

### 4.1. Identification of the closed polyhedral cells

First, the topology/connectivity of the network is determined by identifying the vertices and edges. Then, a search for the shortest possible, planar, convex loops around each node in the network identifies the faces. Finally, the relationship between vertices, edges and faces are stored in a winged-edge data structure [9]. A breath-first-search (BFS) algorithm is used to find the faces belonging to the same polyhedral cell.

### 4.2. Initial form polyhedron

In this step, we construct a form polyhedron that is topologically the same as the desired final form polyhedron, but simply place its one-valent vertices at the centroids of the external faces of the force polyhedron and the other vertices at the centroid of each corresponding cell. This means that the edges of the form polyhedron will generally not be perpendicular to the faces of the force polyhedron.

First, we construct the adjacency list of the polyhedral cells, and compute the centroid of each cell. The centroid of each cell is then connected to its immediate neighbours in the adjacency list. Finally, the centroid of each cell is connected to the cell's external faces, if any exists. A face is external if it is not shared with any other cell.

### 4.3. Imposing perpendicularity

In order for the form and force polyhedrons to be reciprocal, the edges of the form polyhedron should be perpendicular to the corresponding faces in the force polyhedron.

We impose perpendicularity through as iterative procedure in which all iterations consist of two steps. At the start, we compute the normal vectors of the faces of the force polyhedron. Then, in the first step of all iterations, we rotate the edges such that they are parallel to the normal vectors of the faces. This requires the edges of the form polyhedron to be disconnected. Therefore, in the second step of all iterations, we reconnect the edges, which then results in a form polyhedron that is 'slightly more perpendicular' to the force polyhedron. The procedure is repeated until all the edges are perpendicular to their reciprocal faces up to a chosen tolerance.

Finally, we visualize the distribution of forces by adding thickness to the edges of the form polyhedron, proportional to the area of the reciprocal faces.

**I. Identification of closed polyhedral cells**

**Input:** force polyhedron drawing (Network of connected lines)

Steps
1. Identify vertices and edges.
2. Identify faces.
3. Construct winged-edge data structure.
4. Identify faces belonging to a single cell.

**Output:** closed polyhedrons (cells)

**II. Extract form topology**

**Input:** Closed polyhedrons

Steps
1. Compute adjacency of the cells.
2. Compute cell centroids.
3. Connect centroids of adjacent cells.
4. Connect the centroid of each cell to its external faces.

**Output:** Initial form polyhedron

**III. Impose perpendicularity**

**Input:** Initial form polyhedron

Steps
1. Calculate normal vectors of the faces of the force polyhedron.
2. Repeat until convergence
    1. Rotate edges of form polyhedron to become parallel to normal vectors.
    2. Reconnect edges of the form polyhedron
3. Assign the area of each face of the force polyhedron as thickness to its perpendicular edge in form polyhedron.

**Output:** The reciprocal form polyhedron.

Table 1. Simplified algorithm of form finding and its major steps.

## 5. Applications in design and analysis of spatial structures

To illustrate the versatility of the technique developed in this paper, we applied it to the design of 1) a tree structure, 2) a polyhedral frame with non-parallel loads, and 3) a cellular structure.

### 5.1. A tree structure

In this example, the force polyhedron consists of five polyhedral cells. Three of these cells are five-sided polyhedrons and two of them are tetrahedrons (Fig. 7 top). Therefore, as seen in Figure 7, the reciprocal form polyhedron of this force polyhedron is a tree structure that has two five-valent and three four-valent nodes. Note that the area of the largest face corresponds to the lower branch of the three to which all the loads from the upper branches are transferred (Fig. 7 bottom).

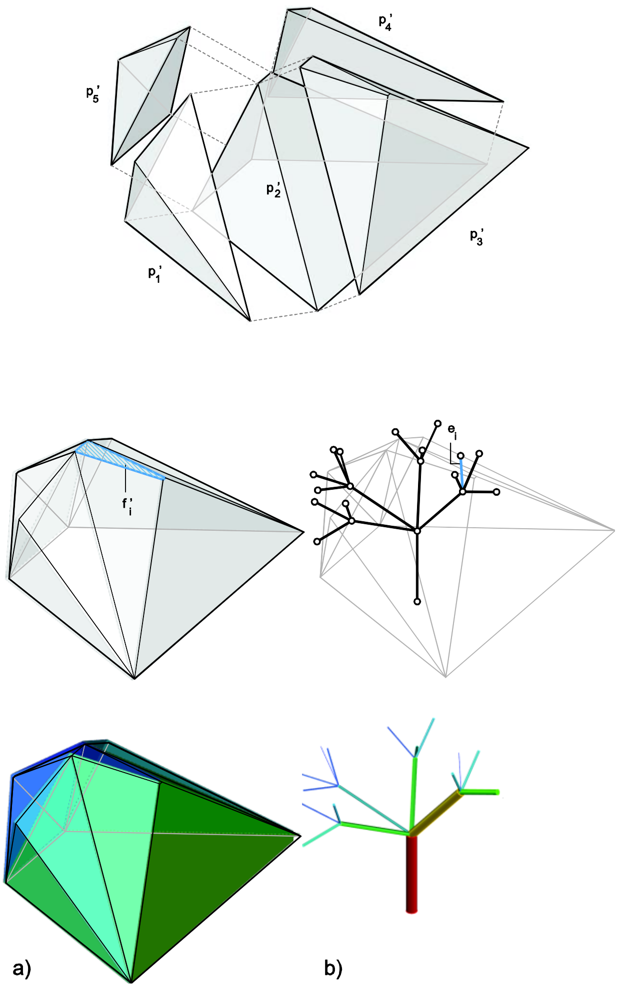


Figure 7. top) Five different polyhedral cells of a force polyhedron; bottom. a) Force polyhedron and its reciprocal tree structure; b) Tree structure reciprocal to the force polyhedron.

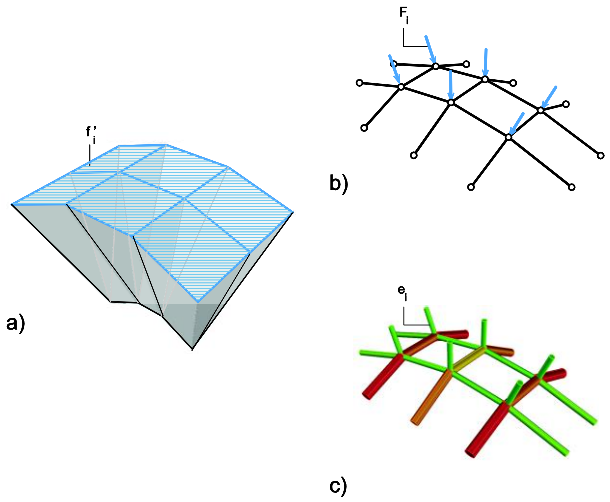


Figure 8. Force polyhedrons as groups of wedges and its reciprocal polyhedral frame; a) The force polyhedrons; b) Polyhedral frame with non-parallel loading; c) The magnitude of the forces represented by tubes with various thicknesses.

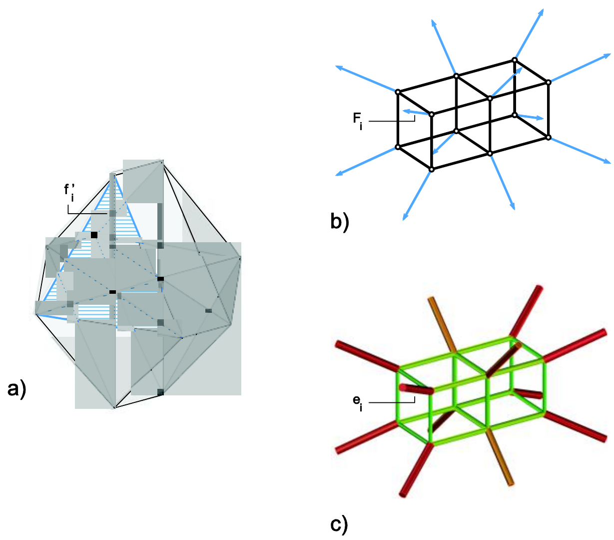


Figure 9. A force polyhedron and its reciprocal, cellular form polyhedron.

### 5.2. A spatial frame with non-parallel loads

Another application of this technique is the design (and analysis) of curved, spatial frames with non-parallel loads. Figure 8 represents a force polyhedron that has a reciprocal structure that can be interpreted as a frame with non-parallel external loads. Similar to the other examples, the areas of the faces are proportional to the magnitude of the forces in the frame. In this example, the areas of the top faces represent the magnitude of the external forces with which the frame is in equilibrium.

### 5.3. A cellular structure

Finally, figure 9.a represents a force polyhedron that has a cellular reciprocal form diagram with non-parallel external loads (Fig 9.b). This form diagram can be interpreted as two polyhedron cells under tensile or compressive external forces.

## 6. CONCLUSION

This paper clarified and illustrated the proposition of Rankine’s reciprocity of form and force diagrams in three dimensions. In addition, it presented clear explanations and illustrations for Maxwell’s procedures for constructing reciprocal diagrams in three dimensions in special cases.

We established a definition of reciprocity for form and force polyhedrons that allows including the external forces in three dimensions.

We presented the computational procedures for constructing a form polyhedron from a given force polyhedron, and described the major steps in the form-finding process in more detail.

We implemented the presented approach in a CAD environment and used this implementation for the design of a tree structure, a spatial frame, and a cellular structure, starting from a chosen force polyhedron in each case.

We used the technique provided in this paper in an interactive environment to explore, and clarify the effects of the changes of the force polyhedrons on the geometry, and the distribution of forces in the form polyhedrons.

The next step in this research is to solve the problem in the reversed direction.

## 7. REFERENCES


[1] Bow, R. H., Economics of construction in relation to framed structures. London: Spon, (1873)

[2] Rankine M., *Principle of the Equilibrium of Polyhedral Frames*, Philosophical Magazine, Vol. 27, Nr. 180 (1864) pp. 92

[3] Maxwell J.C., *On reciprocal figures, frames and diagrams of forces*, Philosophical Magazine, Vol. 27, Nr. 180 (1864) pp. 250-261

[4] Culmann K., Die graphische Statik, Zürich: Verlag Meyer & Zel1er, (1864/1866)

[5] Varignon, P., Nouvelle mécanique ou statique, Paris, (1725)

[6] Cremona, L., Graphical Statics: Two Treatises on the Graphical Calculus and Reciprocal Figures in Graphical Statics, Translated by Thomas Hudson Beare, Oxford: Clarendon, (1890)

[7] Press.Föppl, A., Das Fachwerk im Raume. Leipzig: Verlag von B.G. Teubner, (1892)

[8] Block P., *Thrust Network Analysis: Exploring Three-dimensional Equilibrium*, PhD diss., MIT, Cambridge, MA, USA. (2009)

[9] Baumgart, B.G., *A polyhedron representation for computer vision*, Proceedings National Computer Conference, AFIPS, (1975), pp. 589–596.