A Framework for Comparing Form Finding Methods

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Summary

This paper discusses existing form finding methods in the context of a research project on fabric formworks. The *stiffness matrix*, *force density, surface stress density* and *dynamic relaxation* form finding methods are discussed by mathematically structuring and presenting them in the same way. Based on this, a single computational framework using a sparse branch-node data structure is presented. It is shown how each method defines the internal forces and stiffness of the network, both for linear elements and triangular surface elements, and which solver is used. This single framework marginalizes any differences related to operating platforms, programming language and style, offering a better baseline for comparison of performance. As an example, the minimal surface of a cable-net is calculated, followed by a comparison of the time and iterations required per method. Implications of these results and the framework itself are discussed.

Keywords: *form finding; force density; dynamic relaxation; fabric formwork; branch-node matrix*

1. Introduction

The shape of form-active structures is not known a priori as it depends on the forces and thus requires a process called *form finding* or *shape finding*. Research into computational form finding took place as early as the 1960's. Important strides in computational form finding were made in the 70's and served structural designs such as that of the seminal Munich Olympic stadium's cable-net roof. Cable-net and tensioned membrane roofs, and later also pneumatic structures, (grid-) shells and tensegrity systems, prompted development of more refined methods. A recent application for form finding is not the design of a structural system but that of a construction method: fabric formwork technology [1]. This method has at its heart the concept of casting concrete or rammed earth in prestressed or hanging fabrics. However, computational form finding methods have not yet been specifically developed for fabric formworks, forming the background for this research.

1.1 Existing methods and comparisons

Existing form finding methods can be subdivided in three main families: *force density* methods, *dynamic relaxation* methods and *stiffness matrix* methods. Force density methods refer to all methods that use the concept of the ratio of force to length (or stress to surface area) as a central unit in the calculations. Dynamic relaxation methods use the analogy with motion, where residual forces are converted to velocities and the mass of the nodes determines acceleration. Stiffness matrix methods use real material stiffness matrices in the calculations. This last category may be the least well-defined, with no consensus on name and principal sources. Similar classifications are *non-linear network computation* [2], *computer erecting* [3], *Newton-Raphson Iteration* [4], *non-linear displacement analysis* [5] and *transient stiffness* [6]. All these terms commonly refer to at least one reference by Haug et al., published in the period 1970-1972, and Argyris, Angelopoulos et al., published in the period 1970-1974.

Some comparative overviews of form finding methods can be found [3-5], but these are mostly qualitative in nature and have become somewhat dated. Barnes [4] does compare the storage requirements of dynamic relaxation and stiffness matrix methods per iteration and quotes required numbers of iterations, concluding the dynamic relaxation to be favourable. Lewis [6,7] compared the computational cost of the same two methods for various configurations of cable nets. The conclusion was that the stiffness matrix method did not converge for one of the examples and that dynamic relaxation had lower total computational cost for examples with more degrees of freedom.

Despite almost half a century of literature on form finding methods, thorough comparisons remain rare. Subsequently, it is generally unclear to what extent these methods differ and in which cases one may be preferable over another, specifically whether certain form finding methods are more suited to the particularities of fabric formwork. Compounding this problem is the divide between researchers focusing on particular methods, in spite of them setting similar goals. Comparison is not straightforward as a variety of nomenclatures, mathematical structuring and notation is used.

2. Formulation of a single framework

As a basis for comparison, a single framework using a sparse branch-node data structure, as used in [2], is presented for the three form finding methods. Their implementation in the framework is based on selected, seminal references (force density (FD) [2, 8], dynamic relaxation (DR) [4,9] and stiffness matrix (SM) [10-12]). The mathematical notation used in this paper is as follows: italics represent *scalars*, bold lower-case letters represent **vectors** and bold upper-case letters represent **MATRICES**. Where applicable, the formulation is given for linear elements on the left hand side (e.g. for cable nets) and triangular surface elements on the right hand (e.g. for membrane structures).

Assuming Newton-Raphson's method as the non-linear solver, one iteration in all three methods can be defined as:

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \left(\mathbf{C}^T\mathbf{K}\mathbf{C}\right)^{-1}\left(\mathbf{p} - \mathbf{C}^T\mathbf{L}^{-1}\mathbf{F}\mathbf{C}\mathbf{x}_t - \mathbf{C}^T\mathbf{L}^{-1}\mathbf{F}\mathbf{C}_f\mathbf{x}_{t;f}\right) \quad (1)$$

where **x** are the coordinates, **K** are the branch stiffnesses, **p** are the external loads, **L** are the branch lengths and **F** are the forces. The branch-node matrices **C** and $\mathbf{C}_f$ contain the connectivity of the network of either *m* branches or *f* triangular faces, with *n* free nodes and n_f fixed nodes respectively. Each free node has three degrees of freedom (dof's), one for each direction. Table 1 shows the size of the branch node matrices **C** in the equation, depending on the method and element type used.

Next, the background of Equation 1 is discussed (in 2.1) followed by its input: the stiffness **K** (in 2.2) and the ratio of forces to length $\mathbf{L}^{-1}\mathbf{F}$ (in 2.3). Finally in 2.4, it is discussed how Equation 1 changes when using solvers other than Newton-Raphson's method.

2.1 Static equilibrium

A general network, not in equilibrium, has residual forces **r** (resultants on each node), defined as the difference between the internal forces **f** and external loads **p**.

$$\mathbf{r} = \mathbf{p} - \mathbf{f} \quad (2)$$

Table 1: Size of the branch-node matrices depending on method and use for multiplication with ***F*** *or* ***K***

Type of element	Forces **F**	Stiffness **K**	Stiffness **K**
method	DR, FD, SM	DR, FD	SM
linear	(m,n)	(m,n)	$(3\cdot m,3\cdot n)$
triangular	$(3\cdot f,n)$	$(3\cdot f,n)$	$(9\cdot f,3\cdot n)$

When $\mathbf{f} = \mathbf{p}$ there is equilibrium. Therefore, any solving strategy minimizes the residual forces $\mathbf{r}$ to achieve static equilibrium. Applying Hooke's Law for the residual forces, the stiffness $\mathbf{K}$ is introduced. Note that $\mathbf{K}$ is the derivative of the forces $\mathbf{r}$ as a function of the displacements $\Delta\mathbf{x}$.

$$\Delta\mathbf{x} = \mathbf{K}^{-1} \cdot \mathbf{r} = \mathbf{K}^{-1} \cdot (\mathbf{p} - \mathbf{f}) \quad (3)$$

With the known geometry at time step t, the new geometry will be:

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \Delta\mathbf{x} = \mathbf{x}_t + \mathbf{K}^{-1}(\mathbf{p} - \mathbf{f}) \quad (4)$$

Equation 4 is the same as Equation 1, but Equation 1 clarifies how the entire network can be computed through simple matrix operations using the branch-node matrices.

2.2 Stiffness K

The stiffness $\mathbf{K}$ of the elements is the sum of the elastic stiffness $\mathbf{K}_e$ and geometric stiffness $\mathbf{K}_g$ and shown for linear [10] and triangular elements [12]. In the case of stiffness matrix methods:

$$\mathbf{K}_{e(3,3)} = \frac{EA}{L_0}\mathbf{G}^T\mathbf{G} \quad (5a) \qquad \mathbf{K}_{e,\Delta(9,9)} = A_\Delta t \cdot \mathbf{T}^T\mathbf{D}\mathbf{T} \quad (5b)$$

$$\mathbf{K}_{g(3,3)} = \frac{F}{L}\left(\mathbf{I}_3 - \mathbf{G}^T\mathbf{G}\right) \quad (6a) \qquad \mathbf{K}_{g,\Delta(9,9)} = \bigoplus_{i=1}^{3}\mathbf{K}_{g,i} \text{ ; i=side 1,2,3} \quad (6b)$$

where EA is the branch stiffness, L_0 is the initial branch length, $\mathbf{I}_3$ is a diagonal identity matrix, A_Δ is the triangle surface area, t is the triangle thickness, $\mathbf{D}$ is the plane stress constitutive matrix and $\mathbf{T}$ is a transformation matrix. The transformation matrix $\mathbf{T}$ consists of multiple matrices for which the reader is referred to either [12] or [13]. $\mathbf{G}$ is a matrix of the direction cosines, which are the ratios of the three coordinate differences $\mathbf{u}$ to the length L of each branch.

The computational cost of the stiffness matrix $\mathbf{K}$ can be reduced by lumping the stiffness [9, 14]. This results in a single value per node, giving us the stiffness $\mathbf{K}$ for the dynamic relaxation method:

$$K_{g(1,1)} = \frac{EA}{L_0} \quad (7a) \qquad \mathbf{K}_{e,\Delta(3,3)} = \frac{t}{4A_{\Delta,0}} \cdot D_{(1,1)} \cdot \mathbf{I}_3 \quad (7b)$$

$$K_{g(1,1)} = \frac{F}{L} \quad (8a) \qquad \mathbf{K}_{g,\Delta(3,3)} = \frac{t}{4A_\Delta} \cdot 2 \cdot \sigma_0 \cdot \mathbf{I}_3 \quad (8b)$$

where σ_0 is the isotropic stress. To even further simplify the stiffness, the elastic stiffness can be disregarded entirely while assuming a constant value for the force F or for the geometric stiffness F/L. The latter is known as the force density Q [2] or constant tension coefficient [4] as with force and surface stress density methods:

$$K_{(1,1)} = \frac{F}{L} = Q = \text{const.} \quad (9a) \qquad \mathbf{K}_{(3,3)} = \frac{\sigma}{A_\Delta}\mathbf{I} = Q_\Delta\mathbf{I} = \text{const.} \quad (9b)$$

2.3 Element forces F

Form finding requires the prescription of forces and/or unstressed lengths. The force F, based on the elastic stiffness EA, unstressed length L_0 and pretension F_0 can be defined in the case of stiffness matrix [11] and dynamic relaxation methods [9]:

$$F_{(1,1)} = \frac{EA}{L_0}\left(L - L_0\right) + F_0 \quad (10a) \qquad \mathbf{F}_{\Delta(3,3)} = \mathbf{D}\boldsymbol{\varepsilon} + \bigoplus_{i=1}^{3}\frac{\sigma_0}{\tan\alpha_i} \quad (10b)$$

where **ε** is the strain formulation for the triangle [13]. Alternatively, using a constant force density [2], the forces are:

$$F_{(1,1)} = Q \cdot L \quad (11a) \qquad \mathbf{F}_{\Delta(3,3)} = \bigoplus_{i=1}^{3} \frac{\sigma_0}{\tan \alpha_i} = \bigoplus_{i=1}^{3} \frac{Q_\Delta \cdot A_\Delta}{\tan \alpha_i} \quad (11b)$$

When the lengths L_0 are unknown, they are calculated by combining Equations 10a and 11a.

$$L_0 = \frac{EA \cdot L}{Q \cdot L + EA - F_0} \quad (12)$$

The original force density formulation [2] is obtained by combining Equations 1, 9a and 11a and setting $\mathbf{D}=\mathbf{C}^{\mathrm{T}}\mathbf{QC}$ and $\mathbf{D}_{\mathrm{f}}=\mathbf{C}^{\mathrm{T}}\mathbf{QC}_{\mathrm{f}}$:

$$\mathbf{x}_{t+1} = \mathbf{x}_t + (\mathbf{D})^{-1}(\mathbf{p} - \mathbf{D}\mathbf{x}_t - \mathbf{D}_f\mathbf{x}_{t;f}) = (\mathbf{D})^{-1}(\mathbf{p} - \mathbf{D}_f\mathbf{x}_{t;f}) \quad (13)$$

2.4 Solvers

Stiffness matrix methods typically use Newton-Raphson iterations (Equation 1), or modified procedures in which the initial stiffness matrix is held as constant. When using the modified procedure as well as when using the lumped stiffness in dynamic relaxation, **K** is no longer the derivative of the residual forces **r** as a function of the displacements $\Delta\mathbf{x}$, and therefore the solver is more akin to the simpler Euler method. Dynamic relaxation additionally introduces a two-step method known as Leapfrog integration, where information from the previous iteration is retained and reused. Introducing the velocity **v** and step size h, Equation 4 becomes:

$$\mathbf{v}_t = \mathbf{v}_{t-1} + h \cdot (\mathbf{K})^{-1}(\mathbf{p} - \mathbf{f}) \quad (14)$$

$$\mathbf{x}_{t+1} = \mathbf{x}_t + h \cdot \mathbf{v}_t \quad (15)$$

The simple Euler method can be extended in three ways: using information of the derivatives such as with Newton-Raphson (multiderivative), using information from multiple previous iterations as with Leapfrog (multistep), and/or including multiple calculations per iteration to refine the prediction for the next step (multistage). A form finding method strongly related to dynamic relaxation, called the *particle spring* system [15], implements such a multistage method, a classic fourth-order Runge Kutta [16] for reasons of stability.

3. Minimal surface

As an example of the framework presented here, the stiffness matrix, dynamic relaxation with viscous and kinetic damping [9] and force density methods are compared by form finding a hyperbolic paraboloid cable-net (Figure 1) with fixed boundaries. Three mesh sizes are computed, with an increasing number of degrees of freedom (3 per node): n=3x3, 27 dof's (Figure 1), n=7x7, 147 dof's and n=15x15, 675 dof's. The form finding starts from an initially flat net (Table 2).

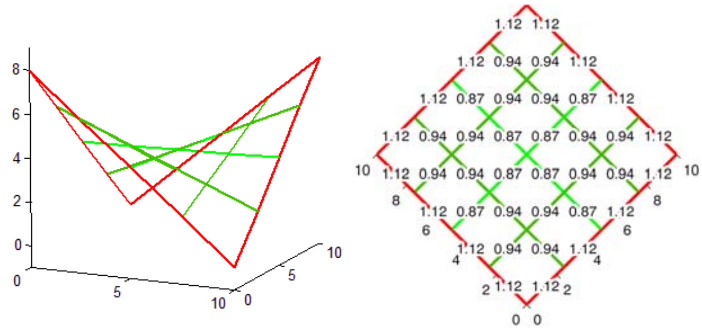


Fig. 1: Normalized forces $\mathbf{f}_{norm}$ *for Q=1, or EA=F_o=1*

As a first trial, trivial input parameters are used: Q=1, EA=F_0=1 and all unstressed lengths L_0 are equal. The solution is considered to have converged when the squared sum of all residual forces changes by less than 1% between iterations, or when the residual forces are 0. The forces in the results are normalized to their average for comparison ($\mathbf{f}_{\mathrm{norm}}=\mathbf{f}/F_{\mathrm{avg}}$).

With these particular input values, the resulting force distribution of all three methods was found to be identical (Fig. 1). In this case, the force density method remains linear and solves the problem the fastest, after one 'iteration'.

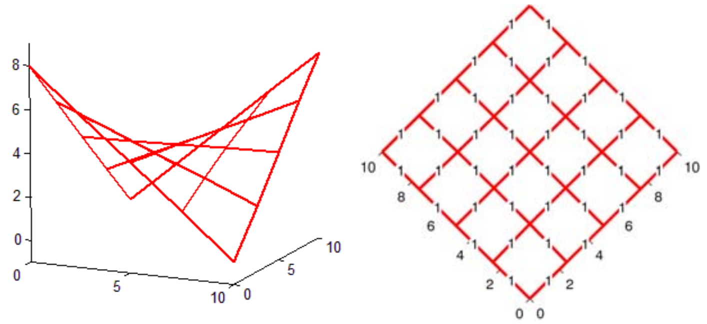
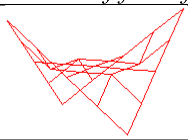
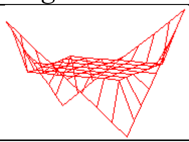
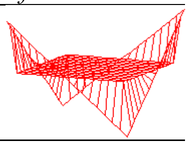
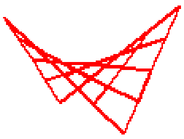
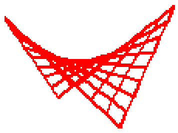
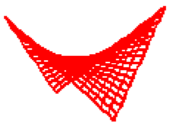


Fig. 2: Normalized forces $\mathbf{f}_{norm}$ *for Q=1, or F=1 and EA=0*

Table 2: Duration of form finding and number of iterations

Initial net			
Result	 dof's = 27	 dof's = 147	 dof's = 675
Method	time t_{norm} (iterations)	time t_{norm} (iterations)	time t_{norm} (iterations)
SM	1.66 (10)	13.2 (17)	29.8 (63)
$DR_{vis.}$	1.23 (11)	**1.00** (19)	1.35 (91)
$DR_{kin.}$	**1.00** (15)	1.67 (25)	**1.00** (39)
FD	1.07 (10)	2.40 (13)	3.20 (35)
t_{min} (s)	**0.003**	**0.006**	**0.15**

In a second calculation, the goal was to find a minimal surface by specifying a constant force throughout the network. The input parameters are: F=1 and EA=0. Initially the same convergence criterion is chosen. The smallest squared length $\mathbf{L}^T\mathbf{L}$ (the most minimal surface) found is then used as a new convergence criterion to make sure that each result is equal in lengths and has uniform force distribution (Fig. 2). The average duration to converge for 100 trials per method is normalized to the minimum for comparison ($\mathbf{t}_{norm}=\mathbf{t}/t_{min}$).

Dynamic relaxation performs increasingly better at more degrees of freedom. Note that the stiffness matrix method also becomes more unstable requiring increasing use of controls as discussed in [10]. The lesser performance of dynamic relaxation with viscous damping in most, but not all cases, seems to be related to the influence of the initial net (the relative length of the edge branches differs for each of the three meshes) and the greater stability that is commonly attributed to kinetic damping. The relative difference in time between stiffness matrix and dynamic relaxation methods compares well with results from Lewis [7] (factor 2.2 at 36 dof's), however in her case viscous damping performs better and the number of iterations is much higher. These differences can be attributed to her example starting from a different net (less susceptible to instability), and having an external load p and $EA\neq0$.

4. Conclusions

The single framework presented here allows the direct comparison of different form finding methods. The mathematical formulation provides insight in how the internal forces and stiffness of the network are calculated with each method and which parameters can be used to control the calculations. This in turn can be used to decide which method is most appropriate for particular applications and goals. The framework offers a basis for extending these methods to other element types, material models and solvers. It also allows hybrid solutions combining strengths of different methods.

An example of the results that this framework can generate demonstrates which method is most suitable for a particular case in terms of the necessary computational time. Further application to other situations will provide information on the general convergence and stability of these methods.

This framework has been created in the context of development of an appropriate form finding method for fabric formwork technology. Future research will focus on applying this framework to the particulars of fabric formwork, for example by evaluating form finding methods when incorporating aspects such as external concrete pressures, fabric wrinkling and sliding support conditions. More generally, this framework can provide new research in form finding a starting point for assessment of existing methods and the benchmarking of new methods.

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